Numerical Note: Solving Housing Macroeconomic Models with Discrete Choices*

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The latest version is available here
The associated code is here

Abstract

In this note, I describe how to solve housing models à la Sommer and Sullivan (2018) with non-convex housing costs. The presence of discrete house size choice makes the value functions discontinuous and generates kinks. In those models, first order conditions (FOC) are still *necessary*, but not sufficient. I use a version of the DC-EGM algorithm developed by Iskhakov et al. (2017) to solve the model. It combines fast standard Endogenous Grid Method (EGM) with a search around the best solutions of the FOC using an upper envelope condition. The resulting algorithm is fast and equilibrium is found in few minutes. The baseline economy and the policy experiment of repealing the mortgage interest deduction produce results similar those found in Sommer and Sullivan (2018).

Keywords: Housing, Endogenous grid method, Discrete choices, DC-EGM.

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In this note I provide details of the algorithm used to solve standard housing macroeconomic models à la Gervais (2002) with endogenous rent and house price as in Sommer and Sullivan (2018)[1]. The model is identical to the one used in the latter paper and the results produced by my code are similar. To solve the model, I use a version of the algorithm developed in Iskhakov et al. (2017)[2]. The algorithm finds optimal policy functions in 3 to 4 seconds, simulates the model in less than 0.1 seconds and finds equilibrium prices in few minutes. The program is written in C and can be downloaded here: `https://github.com/AGaillardTSE/housing`.

1 Model

In this section, I quickly recall the dynamic problem of the household. More details are available in Sommer and Sullivan (2018). For simplicity, I consider only the stationnary equilibrium and abstract from the transitional dynamics.

Time is discrete. There is a mass one of household in the economy. All individuals are characterized by an idiosyncratic income shock $y \in \mathcal{Y}$, asset level $k \in \mathcal{K}$ and house size on a discrete grid $h \in \mathcal{H} = \{0, h_1, ..., h_H\}$. The utility flow of an household is given by:

$$V = \sum_{t=0}^{\infty} u(c_t, s_t)dt = \sum_{t=0}^{\infty} \frac{(c_t^{\alpha} s_t^{1-\alpha})^{1-\sigma}}{1-\sigma} dt$$

where c_t and s_t stand respectively for the consumption of goods and housing service. Housing service can take values on the following discrete grid $s \in \mathcal{S} = \{s_0, s_1, ..., s_S\}$, where $s_0 > 0$. The dynamic problem of such a household is given by:

$$\begin{aligned} V(k,h,y) = \max_{c,s,h',k'} \; & u(c,s) + \beta \sum_{y' \in \mathcal{Y}} \pi_y(y'|y) V(k',h',y') \\ s.t \quad & c \leq wy(1-\tau(wy)) + (1+r^m)\mathbb{1}_{k<0}k + (1+r^d)\mathbb{1}_{k\geq 0}k + q(h-h') + \rho(h'-s) \\ & \quad - \tau_h qh' - \delta_h qh' - \phi \mathbb{1}_{h'>s} - \mathbb{1}_{h'\neq h}(\tau_s qh + \tau_b qh') - k' \\ & k' \geq -(1-\theta)qh' \\ & h' \geq s \quad \text{if} \quad h' > 0 \qquad\qquad c > 0 \end{aligned}$$

where τ is a progressive tax function, r^m and r^d are respectively the interest rate on mortgage and deposit, q and ρ define the house price and the rent, τ_h, τ_s and τ_b are respectively the property tax rate and the tax rates applied on the sale value of the previous house size h and on the purchase value of the current house size h'. Finally, δ_h is the maintenance cost on the current house h' and ϕ is a fixed cost paid by a landlord (i.e. $h' > s$).

[1] The code of the original paper can be found here: `https://www.aeaweb.org/articles?id=10.1257/aer.20141751`.

[2] Their code can be found here: `https://github.com/fediskhakov/dcegm`.

2 Numerical solution

The presence of discrete choices over housing service s and house size h' makes the problem non-standard. In particular, it generates kinks in the value function. I use a version of the Iskhakov et al. (2017) algorithm is order to get rid of the generated kinks. It uses standard EGM with an upper envelope condition that removes suboptimal solutions in the generated endogenous grid.

2.1 Policy functions

Grid definition In contrast to Sommer and Sullivan (2018), I create a semi-endogenous grid for assets that depends on the current house price q and the chosen house size h', based on the downpayment requirement (DRP) constraint. Here, the endogenous grid is different from the one that I will consider later for EGM. The standard exogenous grid over *end-of-period* asset level now depends on q and h'. I call it semi-endogenous since the construction is fixed (i.e. equispaced), but the asset values in the grid change with h' and q. By doing so, I am able to consider only grid point that can be actually chosen by the household. Therefore, in constrast to Sommer and Sullivan (2018), for the same number of grid points, all the points are feasible which increases the precision of my algorithm.

Second, I solve value functions after the discrete choices of h' and s have been made. I solve these values on a grid of *cash-on-hand* (coh). This is useful since it reduces the size of the loop over finding the optimal next period asset level k'. As for the grid for assets, the coh is defined on a grid $\mathcal{C}$ that depends on the current price q and the chosen house size h', such that I compute only feasible allocations.

For a given value of q, I construct H semi-endogenous grid of *end-of-period* asset level $\mathcal{G}_k(h', q)$ equispaced between $k_{min}(h', q)$ and k_{max}, with the value of $k_{min}(h', q)$ defined as follows:

$$k_{min}(h', q) = -(1-\theta)qh'$$

Similarly for the coh, I define a sem-endogenous grid for *cash-on-hand* $\mathcal{G}_c(h', q)$ equispaced with c_{max} and $c_{min}(h', q)$, with the minimums defined as follows:

$$c_{min}(h', q) = -(1-\theta)qh' + \epsilon$$

That is, the minimum coh level can not be lower than the downpayment requirement because otherwise, as the value of k' has to satisfy the (DRP) constraint, the consumption would be negative, which is not possible. These semi-endogenous grids guarantee that all computed points of the value function are feasible. In the following, I will simplify the notations and I will use $coh = coh_{i,j}$ and $k' = k'_{i,j}$ for a given q, with $j \in [0, H]$ and $i \in [0, K]$.

EGM Taking the previous *end-of-period* grids as given as well as the value $V(k', h', y')$ computed during the previous iteration[3]. I solve the value functions using standard EGM algorithm. Given the next period value function, I compute an endogenous *cash-on-hand* grid $\hat{\mathcal{G}}_c$ that depends on the *end-of-period* asset level k', the current choices of s and h' and the consumption level obtained from the derivative of the continuation value with respect to k', following the first order conditions, such that:

$$c = u^{-1}\Big(\beta \sum_{y' \in \mathcal{Y}} \pi_y(y'|y) V_k(k', h', y'), s\Big)$$
$$\hat{coh} = k' + c$$

where $\hat{coh}$ is an endogenous grid point in $\hat{\mathcal{G}}_c$.

Upper envelope condition (UEC) I use the algorithm developed in Iskhakov et al. (2017) to get rid of the kinks in the value functions. Notice that I don't use any smoother with IID Extreme Value Type I choice-specific taste shocks here. I directly detect non-concave region by checking if the generated endogenous grid of *cash-on-hand* in the EGM step is monotonically increasing, following the Algorithm 2 in Iskhakov et al. (2017). It requires to check whether $coh_i > coh_{i+1}$. If so, there is a non-concave region and the endogenous grid of coh as to be corrected of the suboptimal points by taking the solution that produces the highest value (upper envelope condition).

Solve on the grid of cash-on-hand Given this endogenous grid and the value function associated to this grid points, I solve a version of the problem using a value function defined on my semi-endogenous grid of $coh \in \mathcal{G}_c$, as defined in the previous paragraph, such that the program writes

$$\nu(coh, h', y, s) = \max_{c,k'} u(c, s) + \beta \sum_{y' \in \mathcal{Y}} \pi_y(y'|y) V(k', h', y')$$
$$s.t \quad c + k' \leq coh$$
$$k' \geq -(1-\theta) q h'$$

To solve for the optimal k' given current choice h' and s, I locate coh in the grid $\hat{\mathcal{G}}_c$. To deal with the borrowing constraint, notice that $\hat{coh}_0$ defines the endogenous *cash-on-hand* value such that k' eats the borrowing constraint. For all $coh < \hat{coh}_0$, the solution to the problem is to set $k' = -(1-\theta)qh'$. For all $coh > \hat{coh}_0$, I locate $coh \in [\hat{coh}_i, \hat{coh}_{i+1}]$, and then interpolate the value function evaluated at i and $i+1$. After this step, I obtain the value function $\nu(coh, h', y, s)$ and the policy function $k'(coh, h', y, s)$ associated to a $coh \in \mathcal{G}_c(h', q)$, for given choices of h' and s.

[3] In the first iteration, I use as a guess for the value function the utility function with $s = h' = h = k' = 0$ for any k. EGM only require that the value function is increasing in k, which is the case here.

Discrete choices s and h' Having the solutions k' for any different level of coh, s and h', I then compute the optimal discrete choices of s and h' given the state (k, h, y). I therefore compute for each household state (k, h, y, h', s) the corresponding $\tilde{coh} = f(k, h, y, h', s)$. If $\tilde{coh} < c_{min}(h', q)$ the allocation is not feasible and I set $-\infty$ for the corresponding value function $V(k, h, y)$ evaluated at the chosen s and h'. If $\tilde{coh} > c_{min}(h', q)$ the allocation is feasible, and I interpolate the corresponding value using the solutions of the previous step[4]. That is, I locate $\tilde{coh}$ in the semi-endogenous grid $\mathcal{G}_c(h', q)$ in order to recover the optimal saving decision and value function. At this stage, I obtain a set of values defined on the state space (k, h, y), $V(k, h, y)$ with corresponding optimal choices $s(k, h, y)$, $h'(k, h, y)$ and $k'(k, h, y)$.

2.2 Simulation

The resulting policy functions $k'(k, h, y)$ and $h'(k, h, y)$ obtained previously and the transition matrix summarizing the evolution of y' given y (i.e. Π_y) can be used to write a transition matrix for the whole economy.

To do so, I locate the decision rule $k'(k, h, y)$ on the grid $\mathcal{G}_k(h', q)$ such that $k'(k, h, y) \in [k_i, k_{i+1}]$ (since k' as no reason to be on a given value in $\mathcal{G}_k(h', q)$). Then, I compute two weights:

$$w_1 = \frac{k' - k_i}{k_{i+1} - k_i} \qquad w_2 = \frac{k_{i+1} - k'}{k_{i+1} - k_i}$$

w_1 is the fraction of "movers" from state k that goes to k_{i+1} and w_2 is the fraction of individuals in k who move to k_i. As $h'(k, h, y)$ is a discrete decision rule, there is no need for interpolating the decision in order to construct a transition matrix. An individual in h choosing h' will move to the state h'. Concerning the productivity shock y, I use the transition matrix Π_y to draw the movement between a state y to another state y'.

For example, a mass λ of individuals is in the state (k, h, y). All these individuals will choose to save $k'(k, h, y)$ and be in a house size $h'(k, h, y)$. Therefore, all the mass will jump to the $h'(k, h, y)$ house size next period. Concerning the saving decision and the next asset level, the mass is splitted between a fraction w_1 that saves k_{i+1} and a fraction w_2 that saves k_i. Finally, the masses λw_1 and λw_2 on (h', k_{i+1}) and (h', k_i) is again splitted in the dimension of y using the probabilities $\pi(y'|y)$. I obtain that the mass λ in (k, h, y) moves to (k_i, h', y') with mass $\lambda w_2 \pi(y'|y)$ and to (k_{i+1}, h', y') with mass $\lambda w_1 \pi(y'|y)$. All these movements are summarized in a transition matrix M that contains all the movement of each individual falling in a given state (k, h, y)[5].

[4]By construction of the grids for coh and k, for each pair of (k, h, y) there exists a feasible set of choices (s, h') which is feasible.

[5]Notice that because individuals who die in Sommer and Sullivan (2018) are reborn with zero net worth, I construct M such that such individuals switch to the state $(0, 0, 0)$ with probability one.

Having written the transition matrix M properly and given a measure $\gamma(k, h, y) \in \Gamma$ of individuals in (k, h, y) (e.g. I take as a first guess a mass one on $(0, 0, 0)$), I simulate the movements of the measure using M in order to obtain the stationary measure of individuals in the economy (i.e. I iterate on the equation $\Gamma = M^T \Gamma$ until convergence of Γ).

2.3 Equilibrium fixed point

Definition 2.1 (Stationary recursive equilibrium)**.** *Given the state vector $\boldsymbol{x} = (k, h, y)$ with $k \in \mathcal{K}$, $y \in \mathcal{Y}$ and $h \in \mathcal{H}$; a stationary recursive equilibrium in this economy consists of a set of value functions $V(k, h, y)$, policy rules with asset holding $k'(\boldsymbol{x})$, consumption $c(\boldsymbol{x})$, house size choice $h'(\boldsymbol{x})$, housing service $s(\boldsymbol{x})$, prices ($q, \rho \in \mathbb{R}$) and a stationary measure over individuals $\Gamma(\boldsymbol{x})$ such that*

- *the allocation choices maximise the agent problem and $\Gamma(\boldsymbol{x})$ is the stationary probability measure induced by decision rules and the probability Π_y.*
- *the rental and housing markets clear.*

$$(\textit{rental market}) \qquad \int_x (h'(\boldsymbol{x}) - s(\boldsymbol{x}))\Gamma(\boldsymbol{x}) = 0 \tag{1}$$

$$(\textit{housing market}) \qquad \int_x h'(\boldsymbol{x})\Gamma(\boldsymbol{x}) = H \tag{2}$$

where H is the housing supply.

Using the results of the simulation part, we can check whether or not the rental and housing markets clear and we can update the prices accordingly. Because rent and house price are very dependent on the fraction of homeowners, landlords and renters in the economy, I compute the prices using a nested bisection method[6].

1. I set up an upper bound (q_{max}) and lower bound (q_{min}) for q and set $q = \frac{(q_{max}+q_{min})}{2}$.
2. Given this q, set a upper bound (ρ_{max}) and lower bound (ρ_{min}) for rent and guess $\rho = \frac{(\rho_{max}+\rho_{min})}{2}$.
3. Compute the optimal policy functions and simulate the economy. If (1) > 0, this means that there are too much landlords in the economy and we need to lower the rent, we set $\rho_{max} = \rho$. If (1) < 0, then set $\rho_{min} = \rho$. Go back to step 2 until $\rho_{max} - \rho_{min} < \epsilon$.
4. Check whether or not the housing market clear. If $\int_{\mathbf{x}} h'(\mathbf{x})\Gamma(\mathbf{x}) > H$, then I increase q and set $q_{min} = q$. If $\int_{\mathbf{x}} h'(\mathbf{x})\Gamma(\mathbf{x}) < H$, then I lower price q and set $q_{max} = q$.
5. Reconstruct the semi-endogenous grid $\mathcal{G}_c(h', q)$ and $\mathcal{G}_k(h', q)$ given the new q.
6. Iterate until $q_{max} - q_{min} < \epsilon$.

[6] I quickly tried with a version where both prices are determined together, but this results in *bang-bang* solutions.

3 Parameterization

The parameters used in the code are similar to those used in Sommer and Sullivan (2018), except for the housing supply H. Indeed, in their paper they use a Monte Carlo method in order to compute the stationary measure of individuals in the economy and therefore use a value for H which is consistent with the number of simulated individuals. Here instead, I use a different method to compute the stationary measure of individuals. I therefore set H to get approximatively the same house price and rent level as in their paper (i.e. I set $H = 1.2$).

The next section display the performance of the algorithm and the resulting moments which are compared to Sommer and Sullivan (2018). I then reproduce the same experiment of repealing the mortgage interest deduction and show that my algorithm generates the results.

4 Results: comparison

In table 1, I compare the results of the above algorithm to the results in Sommer and Sullivan (2018). Prices (q, ρ) are slightly different as well as the share of homeowners, renters and landlords in the economy. This difference could be attributed to the different definition of the grid for assets that I adopted as well as a different level of housing supply. In particular, I use a semi-endogenous grid for assets with 200, 300 or 500 grid points[7]. The other grids (on h, s, and y) are identical to those used in Sommer and Sullivan (2018).

Moment	**Sommer and Sullivan (2018)**	**200 points**	**300 points**	**500 points**
A. Baseline economy				
% of homeowners	65	67.7	67.1	67.3
% of renters	35	32.3	32.9	32.7
% of landlords	10.1	9.4	9.3	9.2
house price (q)	3.052	3.045	3.057	3.058
rent (ρ)	0.248	0.2402	0.2403	0.2411
B. Repealing the mortgage interest deduction				
% of homeowners	70.2	75.1	72.9	73
% of renters	29.7	24.9	27.1	27
% of landlords	6.8	7.2	6.9	7.4
house price (q)	2.925	2.908	2.922	2.923
rent (ρ)	0.249	0.241	0.2421	0.2425

Table. 1. Results of the algorithm as compared to Sommer and Sullivan (2018).

The main message of the paper, however, remains unchanged. That is, repealing the mortgage interest deduction lowers house price, increases rent, increases the fraction of homeown-

[7] I use the same dimension for $\mathcal{G}_c(h', q)$.

ers and lowers the fraction of landlords in the economy.

Finally, in table 2, I provide the computational time needed for each step of the algorithm. There are three main steps: (i) solving the policy functions, (ii) simulate the model, (iii) obtain a fixed point for the equilibrium prices (q,ρ). The step (iii) iterate on step (i) and (ii) with different set of prices until convergence.

Number of grid points	Policy	Simulation	Equilibrium
200	1.9	0.027	360
300	3.5	0.039	608
500	6.9	0.08	960

Table. 2. Computational speed (in seconds) for the main steps of the algorithm. Computations are done on a MacBook Pro with a 2.5GHz quad-core Intel Core i7 processor with 7 threads.

Solving the model requieres approximately 5 - 15 minutes, which is reasonable if one would want to estimate this type of model using SMM to match particular moments in the data. Notice also that there are many ways to improve the computational speed (precomputing expectations, parallelising on more threads, iterate on marginal utilities instead of value functions, etc...)

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