Heterogenous Agent New Keynesian Model

Numerical solutions*

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1 Model

I describe below the numerical solution for an Heterogenous Agent New Keynesian model in the spirit of Kaplan et al. (2016) but with only one asset. The method used is fragmented in two main steps:

- First, we compute the steady-state of the economy.
- Second, we estimate the behavior of the economy only the transition path. It requieres estimating the price path consistent policy functions

2 Steady State economy

2.1 Household

To make things easy, we assume a simple household problem with two state idiosyncratic shocks for income, such that $z_j \in \{z_1, z_2\}$. Second, we assume that the agent i can save only in one type of asset a_i, but he work l_t hours. We adopt a continuous-time approach in line with Achdou

*Corresponding author: alexandre.gaillard@tse-fr.eu. This draft is based on Achdou et al. (2017). All the mathematics behind this come from their work.

et al. (2017). The household problem can be summarized by:

$$\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t, l_t) dt \tag{1}$$

$$u(c_{i,j,t}, l_{i,j,t}) = \frac{\left[c_{i,j,t} - \Psi z_j \frac{l_{i,j,t}^{1+\frac{1}{\Psi}}}{1+\frac{1}{\Psi}}\right]^{1-\gamma}}{1-\gamma}$$

The budget constraint of the individual with asset level a_i is given by:

$$\dot{a}_i = w z_j l_{i,j} + r a_i - c_{i,j} \tag{2}$$

where w and r are determined by equilibrium conditions. To solve the model, we refer to the methode developped by Achdou et al. (2017). That is, our discretized continuous time formulation for numerical solution of this simple household problem is given by

$$(continuous) \qquad \rho v(a_i, z_j) = u(c_{i,j}) + \partial_a v(a_i, z_j)\dot{a}_i + \lambda_j (v(a_i, z_{j'}) - v(a_i, z_j)) + \partial_t v(a_i, z_j) \tag{3}$$

$$(discrete) \qquad \rho v_{i,j}^{n+1} + \frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} = u_{i,j}^{n} + \frac{v_{i+1,j}^{n+1} - v_{i,j}^{n+1}}{\Delta_a}\dot{a}_{i,j}^{+} + \frac{v_{i,j}^{n+1} - v_{i-1,j}^{n+1}}{\Delta_a}\dot{a}_{i,j}^{-} \tag{4}$$

$$+ \lambda_j (v_{i,j'}^{n+1} - v_{i,j}^{n+1})$$

where I replace $v(a_i, z_j)$ by $v_{i,j}$ and index n means iteration n. $\dot{a}_{i,j}^{-}$ and $\dot{a}_{i,j}^{+}$ can be replaced by:

$$\dot{a}_{i,j}^{-} = min\{0, w z_j l_{i,j} + r a_i - c_{i,j}^{B}\}$$

$$\dot{a}_{i,j}^{+} = max\{0, w z_j l_{i,j} + r a_i - c_{i,j}^{F}\}$$

$$c_{i,j}^{B} = u^{-}1\left(\frac{v_{i,j}^{n+1} - v_{i-1,j}^{n+1}}{\Delta_a}\right) + desutil_l$$

$$c_{i,j}^{F} = u^{-}1\left(\frac{v_{i+1,j}^{n+1} - v_{i,j}^{n+1}}{\Delta_a}\right) + desutil_l$$

This allows us to rewrite:

$$(discrete) \qquad \rho v_{i,j}^{n+1} + \frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} = u_{i,j}^{n} + v_{i,j}^{n+1} y_{i,j} + v_{i-1,j}^{n+1} \zeta_{i,j} + v_{i+1,j}^{n+1} x_{i,j} + \lambda_j (v_{i,j'}^{n+1} - v_{i,j}^{n+1})$$

$$(matrix\ form) \qquad \rho v^{n+1} + (v^{n+1} - v^{n})\frac{1}{\Delta} = u^{n} + A^{n} v^{n+1} \qquad A^{n} = B^{n} + \Lambda$$

where we have

$$y_{i,j} = \dot{a}_{i,j}^{-} - \dot{a}_{i,j}^{+} \qquad x_{i,j} = \dot{a}_{i,j}^{+} \qquad \zeta_{i,j} = \dot{a}_{i,j}^{-}$$

Therefore, matrices B^n and C are given by:

$$B^n = \begin{bmatrix} y_{1,1} & x_{1,1} & 0 & \cdots & & & & \\ \zeta_{2,1} & y_{2,1} & x_{2,1} & 0 & & & & \\ 0 & \ddots & \ddots & \ddots & & & & \\ \vdots & 0 & \zeta_{I,1} & y_{I,1} & & & & \\ & & & & \ddots & & & \\ & & & & y_{1,J} & x_{1,J} & 0 & \dots \\ & & & & \zeta_{2,J} & y_{2,J} & x_{2,J} & 0 \\ & & & & 0 & \ddots & \ddots & \ddots \\ & & & & \vdots & \ddots & \zeta_{I,J} & y_{I,J} \end{bmatrix} \quad \Lambda = \begin{bmatrix} 0 & \dots & 0 & \lambda_1 & 0 & \dots & \dots & \\ 0 & \dots & \dots & 0 & \lambda_1 & 0 & \dots & \\ & & & & \ddots & & & \\ \lambda_2 & 0 & \dots & \dots & & & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & & & \dots & 0 \end{bmatrix}$$

2.2 Firms

Based on https://www3.nd.edu/~esims1/new_keynesian_model.pdf and http://www.princeton.edu/~moll/ECO521_2016/Lecture2_ECO521.pdf.

Final good producers There is a representative final goods producer which aggregates a continuum of intermediate inputs indexed by $k \in [0, 1]$, such that:

$$Y = \left(\int_0^1 y_{k,t}^{\frac{\epsilon-1}{\epsilon}} dk \right)^{\frac{\epsilon}{\epsilon-1}}$$

where $\epsilon > 0$ is the elasticity of substitution across goods. Cost minimization implies that demand for intermediate good j is

$$y_{k,t}(p_{k,t}) = \left(\frac{p_{k,t}}{P_t} \right)^{-\epsilon} Y_t \qquad \text{where} \qquad P_t = \left(\int_0^1 p_{k,t}^{1-\epsilon} dk \right)^{\frac{1}{1-\epsilon}}$$

Intermediate goods producers Each intermediate good is produced by a monopolistically competitive producer which use only labor $n_{k,t}$ as input, such that:

$$y_{k,t} = Z_t n_{k,t}$$

where Z_t is an aggregate TFP shock. From cost-minization problem, we have

$$w_t = \frac{\epsilon - 1}{\epsilon} Z_t n_t \tag{5}$$

Such that the profit is equal to:

$$\Pi_t = Z_t n_t \left(1 - \frac{\epsilon - 1}{\epsilon} \right) \tag{6}$$

Aggregation

$$Y = \int_0^1 y_{k,t} = \int_0^1 Z_t n_{k,t} = Z_t L_t^d \tag{7}$$

where, according to market clearing condition, we must have:

$$L_t^d = L_t^s = \bar{z} w^{\Psi}; \tag{8}$$

For firm, we assume that Y is equal to total demand, such that

$$Y = C + \Pi + rB_d \tag{9}$$

2.3 Monetary policy

We assume that the monetary policy adopt a simple taylor rule, such that:

$$i_t = \bar{r}_{ss} + \phi \pi_t$$

Inflation in our economy behave according to the following law of motion, using a Rotemberg [1982] and a quadratic price adjustement cost, we have for price setting:

$$\begin{aligned} \rho\pi &= \frac{\epsilon - 1}{\theta}\Big(\frac{\epsilon}{\epsilon - 1}\frac{w_t}{Z_t} - 1\Big) + \dot{\pi}_t \\ w_t &= \frac{\theta}{\epsilon}(\rho\pi_t - \dot{\pi}_t) + 1 \end{aligned}$$

2.4 Equilibrium

At the equilibrium, bond demand should be equal to supply.

$$B_d = B_s \tag{10}$$

We assume that $B_d = 0.1$. For B_s, we have

$$B_s = \int_0^1 ag(a) \tag{11}$$

Equilibrium implies that $B_d = B_s$, such that interest rate r adjust. Condition (7) holds by Walras law.

3 Transition Dynamics

In the economy, everything is determined, except inflation rate. Therefore, when studying the path between two steady states, we are looking for the path of π_t for which all markets clear. In order to do so, we have to know the initial and the final condition. These two conditions correspond to two steady-states.

3.1 M.I.T shocks

We are looking for an unanticipated shock arriving at data $t = 1$. Therefore, initial condition ($t = 0$) and final condition ($t = 1$) are described by the same steady-state economy. At steady-state, $\pi_t = 0$ and $r = \bar{r}_{ss}$. Our algorithm for transition path is in line with Achdou et al. (2017). The system to be solved is:

$$
\begin{aligned}
(Bond\ market) \qquad & B_d(t) = \int_{\bar{a}}^{\infty} a g_1(a,t) da + \int_{\bar{a}}^{\infty} a g_2(a,t) da \\
(HJB) \qquad & \rho v_j(a,t) = max_c u(c) + \partial_a v_j(a,t)\dot{a}(t) + \lambda_j \Big[v_{-j}(a,t) - v_j(a,t) \Big] + \partial_t v_j(a,t) \\
(Fokker-Plank) \qquad & \partial_t g_j(a,t) = -\partial_a \Big[s_j(a,t) g_j(a,t) \Big] \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t)
\end{aligned}
$$

The algorithm to solve the transition dynamics is the following

1. for iteration l, guess path of π_t^l given that $\pi_0^l = 0$.

2. given π_t^l, compute the associated prices r_t, w_t and solve the HJB equation. To solve HJB, proceed backward. Start at time $T-1$ where v_T are given by v^{ss}. Then, compute v^{T-1}. Do the same for v^{T-1} given v^{T-1}. You get saving decision using the difference between the current and future value functions. Given v^{t+1}, the system to be solved can be summarized by:

$$
\rho v^t = u^{t+1} + A^{t+1} v^t + \frac{1}{\Delta_t}(v^{t+1} - v^t)
$$

   where A^{t+1} is the transition matrix computed above.

3. Given saving behavior, solve the Fokker-Plank equation using initial condition $g_j(a,0) = g_j(a,ss)$. Go forward in time to compute $g_j(a,t)$. To do so, we can use directly the transition matrix A^{t+1}. Such that given the Fokker-plank equation we have:

$$
\frac{g^{t+1} - g^t}{\Delta_t} = A^{t+1} g^t \qquad g^{t+1} = (\mathbf{I} - \Delta_t A^{t+1})^{-1} g^t
$$

4. Given saving behavior and the distribution g, compute

$$
S(t) = \int_{\bar{a}}^{\infty} a g_1(a,t) da + \int_{\bar{a}}^{\infty} a g_2(a,t) da
$$

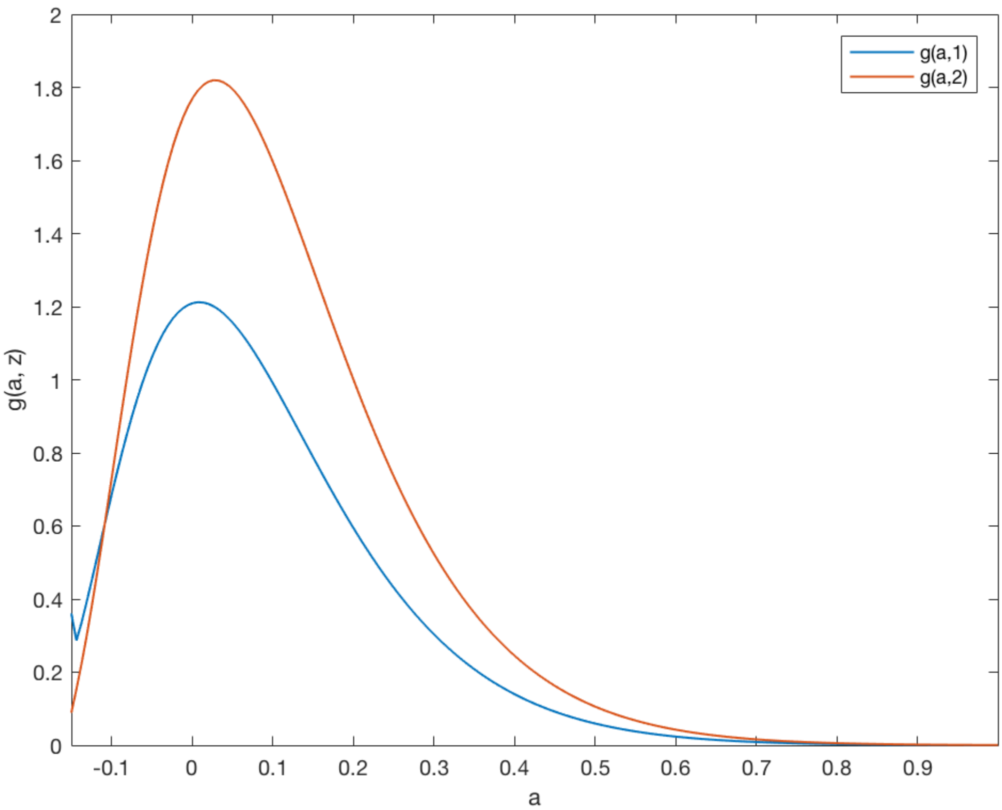
5. update prices $\pi_t^{l+1} = \pi_t^l - \xi \frac{dS(t)}{dt}$, where $\xi > 0$.

6. stop when π^{l+1} is sufficiently close to π^l.

4 Results

4.1 Steady-state distribution

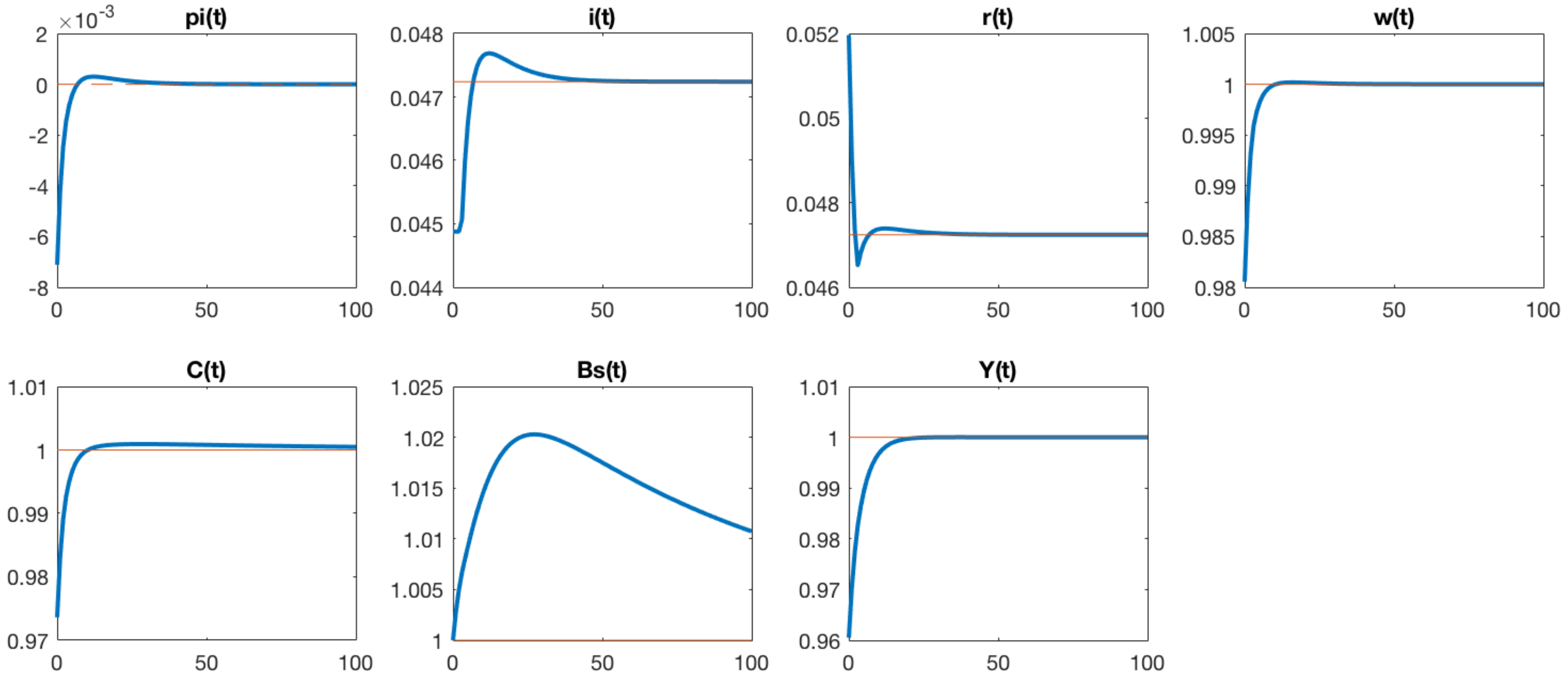
Fig. 1. Steady-state distributions



4.2 MIT shocks

We construct a sequence of TFP shock. At date $t = 1$ we generates a shock which is persistent.

Fig. 2. Steady-state distributions



References

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