Aiyagari Model in Continuous Time with Jump-Drift Income Process*

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1 Model

I model Aiyagari (1994) with a jump-drift process for income as in Kaplan et al. (2016). We therefore divide the income process in permanent-transitory components.

Income process

$$\begin{aligned}
logz &= z_j^1 + z_k^2 \\
dz_j^1 &= -\beta_1 z_j^1 dt + \epsilon_j^1 dN_j^1 \quad \text{with} \quad \epsilon_j^1 \sim \mathcal{N}(0, \sigma_1^2) \\
dz_k^2 &= -\beta_2 z_k^2 dt + \epsilon_k^2 dN_k^2 \quad \text{with} \quad \epsilon_k^2 \sim \mathcal{N}(0, \sigma_2^2)
\end{aligned}$$

where, for example, dN_j^1 is a pure Poisson process with arrival rate λ_1.

HJB equation

$$\begin{aligned}
\rho v(a, z^1, z^2) = max_c u(c) &+ v_a(a, z^1, z^2)\dot{a} + v_{z^1}(a, z^1, z^2)(-\beta_1 z^1) + v_{z^2}(a, z^1, z^2)(-\beta_2 z^2) \\
&+ \lambda_1 \int_x (v(a, x, z^2) - v(a, z^1, z^2))\phi_1(x)dx \\
&+ \lambda_2 \int_x (v(a, z^1, x) - v(a, z^1, z^2))\phi_2(x)dx
\end{aligned}$$

*Corresponding author: alexandre.gaillard@tse-fr.eu. This draft is based on Achdou et al. (2017). All the mathematics behind this come from their work.

2 Numerical solution

I adopt an upwind scheme to solve the model using finite difference equation in line with Achdou et al. (2017). Using implicit method, the model can be rewritten in a discretized version. Let me write $v_{i,j,k} \equiv v(a_i, z_j^1, z_k^2)$ and $\dot{a} = wz + ra - c$, we have:

$$\rho v_{i,j,k}^{n+1} + \frac{v_{i,j,k}^{n+1} - v_{i,j,k}^{n}}{\Delta} = u_{i,j,k}^{n} + v_{a,i,j,k}^{n+1}(we^{z_j^1+z_k^2} + ra_i - c_{i,j,k}) + v_{z^1,i,j,k}^{n+1}(-\beta_1 z_j^1) + v_{z^2,i,j,k}^{n+1}(-\beta_2 z_k^2)$$
$$+ \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j)(v_{i,j',k} - v_{i,j,k}) + \lambda_2 \sum_{k'\neq k}^{\mathcal{K}} \pi_2(k'|k)(v_{i,j,k'} - v_{i,j,k})$$

which can be rewritten

$$\rho v_{i,j,k}^{n+1} + \frac{v_{i,j,k}^{n+1} - v_{i,j,k}^{n}}{\Delta} = u_{i,j,k}^{n} + \frac{v_{i+1,j,k}^{n+1} - v_{i,j,k}^{n+1}}{\Delta_{a,i,F}} s_{i,j,k}^{F} + \frac{v_{i,j,k}^{n+1} - v_{i-1,j,k}^{n+1}}{\Delta_{a,i,B}} s_{i,j,k}^{B}$$
$$+ \frac{v_{i,j+1,k}^{n+1} - v_{i,j,k}^{n+1}}{\Delta_{z^1,j,F}}(-\beta_1 z_j^1)^+ + \frac{v_{i,j,k}^{n+1} - v_{i,j-1,k}^{n+1}}{\Delta_{z^1,j,B}}(-\beta_1 z_j^1)^-$$
$$+ \frac{v_{i,j,k+1}^{n+1} - v_{i,j,k}^{n+1}}{\Delta_{z^2,k,F}}(-\beta_2 z_k^2)^+ + \frac{v_{i,j,k}^{n+1} - v_{i,j,k-1}^{n+1}}{\Delta_{z^2,k,B}}(-\beta_2 z_k^2)^-$$
$$+ \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j)(v_{i,j',k}^{n+1} - v_{i,j,k}^{n+1}) + \lambda_2 \sum_{k'\neq k}^{\mathcal{K}} \pi_2(k'|k)(v_{i,j,k'}^{n+1} - v_{i,j,k}^{n+1})$$

where, for instance, $s_{i,j,k}^F = \max\{0, we^{z_j^1+z_k^2} + ra_i - c_{i,j,k}^F\}$ and the operator $+$ means $\max\{0, x\}$. Using FOC, the consumption forward is equal to:

$$c_{i,j,k}^F = u'^{-1}\Big(\frac{v_{i+1,j,k}^{n} - v_{i,j,k}^{n}}{\Delta_{a,i,F}}\Big)$$

Regrouping all terms with the same v, we get

$$\rho v_{i,j,k}^{n+1} + \frac{v_{i,j,k}^{n+1} - v_{i,j,k}^{n}}{\Delta} = u_{i,j,k}^{n} + v_{i,j,k}^{n+1}\Big[-\frac{s_{i,j,k}^F}{\Delta_{a,i,F}} + \frac{s_{i,j,k}^B}{\Delta_{a,i,B}} - \frac{(-\beta_1 z_j^1)^+}{\Delta_{z^1,j,F}} + \frac{(-\beta_1 z_j^1)^-}{\Delta_{z^1,j,B}} - \frac{(-\beta_2 z_k^2)^+}{\Delta_{z^2,k,F}}$$
$$+ \frac{(-\beta_2 z_k^2)^-}{\Delta_{z^2,k,B}} - \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j) - \lambda_2 \sum_{k'\neq k}^{\mathcal{K}} \pi_2(k'|k)\Big]$$
$$+ v_{i+1,j,k}^{n+1}\frac{s_{i,j,k}^F}{\Delta_{a,i,F}} - v_{i-1,j,k}^{n+1}\frac{s_{i,j,k}^B}{\Delta_{a,i,B}} + v_{i,j+1,k}^{n+1}\frac{(-\beta_1 z_j^1)^+}{\Delta_{z^1,j,F}}$$
$$- v_{i,j-1,k}^{n+1}\frac{(-\beta_1 z_j^1)^-}{\Delta_{z^1,j,B}} + v_{i,j,k+1}^{n+1}\frac{(-\beta_2 z_k^2)^+}{\Delta_{z^2,k,F}} - v_{i,j,k-1}^{n+1}\frac{(-\beta_2 z_k^2)^-}{\Delta_{z^2,k,B}}$$
$$+ \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j) v_{i,j',k}^{n+1} + \lambda_2 \sum_{k'\neq k}^{\mathcal{K}} \pi_2(k'|k) v_{i,j,k'}^{n+1}$$

Defining the following terms:

$$y_{i,j,k} = -\frac{s_{i,j,k}^F}{\Delta_{a,i,F}} + \frac{s_{i,j,k}^B}{\Delta_{a,i,B}} \qquad x_{i,j,k} = \frac{s_{i,j,k}^F}{\Delta_{a,i,F}} \qquad \xi_{i,j,k} = -\frac{s_{i,j,k}^B}{\Delta_{a,i,B}}$$
$$\chi_{j,k} = -\frac{(-\beta_1 z_j^1)^+}{\Delta_{z^1,j,F}} + \frac{(-\beta_1 z_j^1)^-}{\Delta_{z^1,j,B}} - \frac{(-\beta_2 z_k^2)^+}{\Delta_{z^2,k,F}} + \frac{(-\beta_2 z_k^2)^-}{\Delta_{z^2,k,B}} - \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j) - \lambda_2 \sum_{k'\neq k}^{\mathcal{K}} \pi_2(k'|k)$$

$$\tilde{\beta}_j^{1,F} = \frac{(-\beta_1 z_j^1)^+}{\Delta_{z^1,j,F}} \qquad \tilde{\beta}_j^{1,B} = -\frac{(-\beta_1 z_j^1)^-}{\Delta_{z^1,j,B}} \qquad \tilde{\beta}_k^{2,F} = \frac{(-\beta_2 z_k^2)^+}{\Delta_{z^2,k,F}} \qquad \tilde{\beta}_k^{2,B} = -\frac{(-\beta_2 z_k^2)^-}{\Delta_{z^2,k,B}}$$

$$\pi^1_{j',j} = \lambda_1 \pi_1(j'|j) \qquad \pi^2_{k',k} = \lambda_2 \pi_2(k'|k)$$

Then we have:

$$\begin{aligned}
\rho v_{i,j,k}^{n+1} + \frac{v_{i,j,k}^{n+1} - v_{i,j,k}^{n}}{\Delta} = u_{i,j,k}^n &+ v_{i,j,k}^{n+1}\Big[y_{i,j,k} + \chi_{i,j}\Big] + v_{i+1,j,k}^{n+1} x_{i,j,k} + v_{i-1,j,k}^{n+1} \xi_{i,j,k} \\
&+ v_{i,j+1,k}^{n+1} \tilde{\beta}_j^{1,F} + v_{i,j-1,k}^{n+1} \tilde{\beta}_j^{1,B} + v_{i,j,k+1}^{n+1} \tilde{\beta}_k^{2,F} + v_{i,j,k-1}^{n+1} \tilde{\beta}_k^{2,B} \\
&+ \sum_{j' \neq j}^{\mathcal{J}} \pi^1_{j',j} v_{i,j',k}^{n+1} + \sum_{k' \neq k}^{\mathcal{K}} \pi^2_{k',k} v_{i,j,k'}^{n+1}
\end{aligned}$$

We can thus rewrite our system in matrix form:

$$\begin{aligned}
&\rho v^{n+1} + (v^{n+1} - v^n)\frac{1}{\Delta} = u^n + \mathbf{A}^n v^{n+1} \\
&\mathbf{A}^n = B^n + C + \Lambda
\end{aligned}$$

And the matrix are given by (here when k = 1):

$$B_1^n = \begin{bmatrix}
y_{1,1,1} & x_{1,1,1} & 0 & \cdots & & & & & \\
\xi_{2,1,1} & y_{2,1,1} & x_{2,1,1} & 0 & & & & & \\
0 & \ddots & \ddots & \ddots & & & & & \\
\vdots & 0 & \xi_{I,1,1} & y_{I,1,1} & & & & & \\
& & & & \ddots & & & & \\
& & & & & y_{1,J,1} & x_{1,J,1} & 0 & \cdots \\
& & & & & \xi_{2,J,1} & y_{2,J,1} & x_{2,J,1} & 0 \\
& & & & & 0 & \ddots & \ddots & \ddots \\
& & & & & \vdots & \ddots & \xi_{I,J,1} & y_{I,J,1}
\end{bmatrix}$$

$$C_1 = \begin{bmatrix}
\chi_{1,1} & 0 & \cdots & \tilde{\beta}_1^{1,F} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \ddots & \ddots & 0 & \tilde{\beta}_1^{1,F} & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \chi_{1,1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\tilde{\beta}_2^{1,B} & \ddots & \ddots & \chi_{2,1} & \ddots & \ddots & \tilde{\beta}_2^{1,F} & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \chi_{2,1} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \tilde{\beta}_J^{1,B} & \ddots & \ddots & \chi_{J,1} & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \chi_{J,1} & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \chi_{J,1}
\end{bmatrix}$$

$$
\Lambda_1 = \begin{bmatrix}
0 & \dots & 0 & \pi^1_{2|1} & 0 & \dots & 0 & \pi_{3|1} & \dots & \dots \\
\vdots & \ddots & \ddots & \ddots & \pi^1_{2|1} & \ddots & \ddots & \ddots & \pi^1_{3|1} & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\pi^1_{1|2} & 0 & \ddots & \ddots & \ddots & \ddots & 0 & \pi^1_{3|2} & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
$$

3 Deal with dimensionality

As dimensionality could potentially by very huge. We could divide the problem into K subproblem. We therefore use an explicit formulation for dimension k. We have to solve:

$$
\begin{aligned}
\rho v^{n+1}_{i,j,k} + \frac{v^{n+1}_{i,j,k} - v^n_{i,j,k}}{\Delta} = u^n_{i,j,k} &+ v^{n+1}_{i,j,k}\Big[- \frac{s^F_{i,j,k}}{\Delta_{a,i,F}} + \frac{s^B_{i,j,k}}{\Delta_{a,i,B}} - \frac{(-\beta_1 z^1_j)^+}{\Delta_{z^1,j,F}} + \frac{(-\beta_1 z^1_j)^-}{\Delta_{z^1,j,B}} - \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j)\Big] \\
&+ v^n_{i,j,k}\Big[- \frac{(-\beta_2 z^2_k)^+}{\Delta_{z^2,k,F}} + \frac{(-\beta_2 z^2_k)^-}{\Delta_{z^2,k,B}} - \lambda_2 \sum_{k'\neq k}^{\mathcal{K}} \pi_2(k'|k)\Big] \\
&+ v^{n+1}_{i+1,j,k} \frac{s^F_{i,j,k}}{\Delta_{a,i,F}} - v^{n+1}_{i-1,j,k} \frac{s^B_{i,j,k}}{\Delta_{a,i,B}} + v^{n+1}_{i,j+1,k} \frac{(-\beta_1 z^1_j)^+}{\Delta_{z^1,j,F}} \\
&- v^{n+1}_{i,j-1,k} \frac{(-\beta_1 z^1_j)^-}{\Delta_{z^1,j,B}} + v^n_{i,j,k+1} \frac{(-\beta_2 z^2_k)^+}{\Delta_{z^2,k,F}} - v^n_{i,j,k-1} \frac{(-\beta_2 z^2_k)^-}{\Delta_{z^2,k,B}} \\
&+ \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j) v^{n+1}_{i,j',k} + \lambda_2 \sum_{k'\neq k}^{\mathcal{K}} \pi_2(k'|k) v^n_{i,j,k'}
\end{aligned}
$$

where we just replace subscript $n+1$ by n for z^2. Because we now use explicit formulation, we have to worry about Δ. It must be sufficiently low to respect the CFL condition, see Barles and Souganidis (1990). In matrix notation, we have to solve a system of K equation, such that:

$$
\rho v^{n+1}_k + (v^{n+1}_k - v^n_k)\frac{1}{\Delta} = u^n_k + \mathbf{A}^n_k v^{n+1}_k + \lambda_2 \sum_{k'\neq k} \pi(k'|k)(v^n_{k'} - v^n_k) + Q^{+1}_k v^n_{k+1} + Q^{-1}_k v^n_{k-1} + Q_k v^n_k
$$

$$
\mathbf{A}^n_k = B^n_k + C_k + \Lambda_k
$$

with notation:

$$
\begin{aligned}
\theta_k &= -\frac{(-\beta_2 z^2_k)^+}{\Delta_{z^2,k,F}} + \frac{(-\beta_2 z^2_k)^-}{\Delta_{z^2,k,B}} \\
\chi_j &= -\frac{(-\beta_1 z^1_j)^+}{\Delta_{z^1,j,F}} + \frac{(-\beta_1 z^1_j)^-}{\Delta_{z^1,j,B}} - \lambda_1 \sum_{j'\neq j}^{\mathcal{J}} \pi_1(j'|j)
\end{aligned}
$$

where the matrix Q_k^{-1}, Q_k, Q_k^{+1} are given by:

$$Q_k = \begin{bmatrix} \theta_k & 0 & \dots & \dots & \dots & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \theta_k \end{bmatrix} \qquad Q_k^{+1} = \begin{bmatrix} \beta_k^{\tilde{2},F} & 0 & \dots & \dots & \dots & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \beta_k^{\tilde{2},F} \end{bmatrix}$$

$$Q_k^{-1} = \begin{bmatrix} \beta_k^{\tilde{2},B} & 0 & \dots & \dots & \dots & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \beta_k^{\tilde{2},B} \end{bmatrix}$$

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